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# **STATISTICS OF RAIN-RATE ESTIMATES FOR A SINGLE ATTENUATING RADAR**

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December 1976

GODDARD SPACE FLIGHT CENTER  
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ABSTRACT

Statistics are presented for the estimates of rain rate from an attenuating frequency radar. Included in the estimates are the effects of fluctuations in return power and the rain-rate/reflectivity relationship, as well as errors introduced in the attempt to recover the unattenuated return power. In addition to the Hitschfeld-Bordan correction, two alternative techniques are considered. The performance of the radar is shown to be dependent on the method by which attenuation correction is made.

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# STATISTICS OF RAIN-RATE ESTIMATES FOR A SINGLE ATTENUATING RADAR

## I. INTRODUCTION

Meteorological radars that require relatively high spatial resolution or sensitivity, but that are limited as to the antenna size or transmitted power, make desirable the use of higher frequencies. At X-band frequencies and higher, however, the signal may undergo significant attenuation by the intervening rain as it travels to and from the rain-scattering volume of interest. Therefore, the utility of such a system is in part dependent on the accuracy by which the nonattenuated power, as indicative of the rain rate in the scattering volume alone, can be constructed from the measured return power.

To estimate the rain rate,  $R$ , for a single radar system, the reflectivity factor,  $Z_t$ , is required. The latter quantity can be found directly from the radar equation only if the attenuation,  $k$ , is expressible as a function of either the reflectivity factor or some measurable quantity of the radar system. Because a number of such approximations for attenuation are possible, the question as to which rain-rate estimate should be employed reduces to the choice of the approximation that in some way optimizes radar performance. As will be shown, the accuracy of the rain-rate prediction is strongly dependent on the particular approximation adopted.

The procedure given here is to form three different estimates of rain rate that involve the empirical  $(k, Z_t)$ ,  $(Z_t, R)$  relations and the measurable



quantities of the radar system. Treating as random variables the return power from each range bin, as well as certain parameters that relate  $(k, Z_t)$  and  $(Z_t, R)$ , the mean and variance of each of the estimates are then computed as functions of range and number of independent samples. Calibration errors and finite signal-to-noise ratios are also considered.

## II. RAIN-RATE ESTIMATES

From the standard meteorological radar equation, including attenuation effects, the measured return power may be written as [Battan, 1973]

$$P(r) = CZ_t(r) \left[ 10^{-0.2 \int_0^r k(s) ds} \right] / r^2 \quad (1)$$

On using the relation  $k = \alpha Z_t^\beta$ , then,

$$P(r) = CZ_t(r) \left[ 10^{-0.2 \alpha \int_0^r Z_t^\beta(s) ds} \right] / r^2 \quad (2)$$

where  $k$  = one-way attenuation in dB/km

$P$  = measured radar return power averaged over  $N$  independent samples

$r$  = range in km

$C$  = radar calibration constant

For convenience, the auxiliary quantity,  $Z_m$ , is introduced in terms of the defining relation:

$$P(r) = CZ_m(r) / r^2 \quad (3)$$

Or, on combining equations 1 and 2,

$$Z_t(r) = Z_m(r) 10^{0.2 \alpha \int_0^r Z_t^\beta(s) ds} \quad (4)$$

The quantity,  $Z_m$ , may be interpreted as a measured reflectivity factor in the sense that it can be found directly from the measured power and the known values of range and calibration constant.

The reflectivity factor,  $Z_t$ , and rain rate,  $R$ , are usually given in the form of a power law relation that may be written as  $R = aZ_t^b$ . To estimate the rain rate, it is necessary to express  $Z_t$  in terms of the measured quantity,  $Z_m$ ; or, more simply, given the function,  $Z_m = g(Z_t)$  (equation 4), it is required to find the solution of the inverse problem,  $Z_t = g^{-1}(Z_m)$ . If such a solution exists, the rain rate then becomes  $R = a [g^{-1}(Z_m)]^b$ .

To invert equation 4, it can be seen that, if all attenuation effects are neglected, a crude approximation to  $Z_t$  would be

$${}^0Z(r) = Z_m(r) \quad (5)$$

For any higher order approximation, the  $Z_t$  that appears in the exponent of equation 4 is replaced by its preceding approximation. For first- and second-order iterations, this procedure gives

$${}^1Z(r) = Z_m(r) 10^{0.2 \alpha \int_0^r Z_m^\beta(s) ds} \quad (6)$$



and

$$^2Z(r) = Z_m(r) 10^{0.2 \alpha \int_0^r Z_t^\beta(s) ds}$$

or

$$^2Z(r) = Z_m(r) \exp(0.2 (\ln 10) \alpha \int_0^r Z_m^\beta(w) e^{0.2 \alpha \ln 10 \beta \int_0^w Z_m^\beta(u) du} dw) \quad (7)$$

An exact solution for  $Z_t = g^{-1}(Z_m)$  can be found by using a procedure similar to that given by Hitschfeld and Bordan [1954]. Taking the logarithm of both sides of equation 4, differentiating with respect to range, and, finally, using the substitution,  $u = Z_t^{-\beta}$ , yields the first-order linear differential equation,

$$\frac{du}{dr} + \beta u f(r) + K'\beta = 0 \quad (8)$$

where

$$K' = 0.2 \alpha \ln 10$$

$$f(r) = \frac{d}{dr} \ln Z_m$$

then,

$$u(r) = \frac{1}{Z_m^\beta(r)} \left[ c - K'\beta \int_0^r Z_m^\beta(w) dw \right] \quad (9)$$

The condition,  $Z_t(0) = Z_R(0)$ , requires that  $c = 1$ , and therefore,

$${}^{HB}Z_t(r) = Z_m(r) / (1 - K\beta \int_0^r Z_m^\beta(w) dw)^{1/\beta} \quad (10)$$

Approximating the integrals in the foregoing equations by summations over the relevant range bins and substituting the expressions for  $Z_t$  of equations 6, 7, and 10 into the Z-R relation yield the following estimates of rain rate at the  $n^{\text{th}}$  range bin:

$${}^1R_n = aZ_{mn}^b \exp(Kb \sum_{i=1}^n \epsilon_i Z_{mi}^\beta) \quad (11)$$

$${}^2R_n = aZ_{mn}^b \exp(Kb \sum_{i=1}^n \epsilon_i Z_{mi}^\beta \exp(-\beta K \sum_{j=1}^i \epsilon_j Z_{mj}^\beta)) \quad (12)$$

$${}^{H-B}R_n = aZ_{mn}^b / (1 - K\beta \sum_{i=1}^n \epsilon_i Z_{mi}^\beta)^{b/\beta} \quad (13)$$

where

$$\int_0^r Z_m^\beta(s) ds \cong s \sum_{i=1}^n \epsilon_i Z_{mi}^\beta$$

$s$  = range resolution in km

$$\epsilon_i = \begin{cases} 1 & i \neq n \\ 1/2 & i = n \end{cases}$$

$$\epsilon_j = \begin{cases} 1 & j \neq i \\ 1/2 & j = i \end{cases}$$

The estimates of rain rate given in equations 11 to 13 are exact in the sense that statistical fluctuation and offset errors have been excluded. If  $N$  independent pulses are summed over the  $i^{\text{th}}$  range bin, the average return power is a random variable given by [Marshall and Hirschfeld, 1953; Stogryn, 1975]

$$\hat{P}_i = P_i f_i$$

or

$$\hat{Z}_{mi} = Z_{mi} f_i$$

where the quantities,  $P_i$  and  $Z_{mi}$ , are the expectations of the return power and the measured reflectivity factor, respectively. The probability density function of  $f$  is

$$p(f) = \frac{N^N}{(N-1)!} e^{-Nf} f^{N-1} \quad (f > 0) \quad (14)$$

If, in addition, calibration errors are introduced in which  $C = C_{\text{true}} (1 + E)$ , then,

$$\hat{Z}_{mi} = Z_{mi} f_i / (1 + E) \quad (15)$$

Replacing  $Z_{mi}$  in equations 11 through 13 with  $\hat{Z}_{mi}$  results in the following rain-rate estimates:

$${}^1R_n = ca Z_{mn}^b f_n^b \exp(Kbc^\beta \sum_{i=1}^n \epsilon_i Z_{mi}^\beta f_i^\beta) \quad (16)$$

$${}^2R_n = ca Z_{mn}^b f_n^b \times \exp \left\{ Kbc^\beta \left[ \sum_{i=1}^n \epsilon_i Z_{mi}^\beta f_i^\beta \exp (K\beta c^\beta \sum_{j=1}^i \epsilon_j Z_{mj}^\beta) \right] \right\} \quad (17)$$

$${}^{H-B}R_n = ca Z_{mn}^b f_n^b / (1 - K\beta c^\beta \sum_{i=1}^n \epsilon_i f_i^\beta Z_{mi}^\beta)^{b/\beta} \quad (18)$$

where

$$K = 0.2 s \alpha \ln 10$$

$$c = 1/(1 + E) = C_{\text{true}}/C$$

Quantities  $a$  and  $\alpha$  in the above equations are chosen to be independent random variables with variances of  $\sigma_a^2$  and  $\sigma_\alpha^2$  and means equal to their respective true values. Quantities  $b$  and  $\beta$  are assumed to be fixed.

Appendix A describes the behavior of the iterative approximations and their relationship to the Hitschfeld-Bordan solution. Appendix B provides a generalization of the estimates to the case of finite signal-to-noise ratios.

### III. STATISTICS OF THE RAIN-RATE ESTIMATES

The measures of performance for the rain-rate estimates are taken to be the mean and variance. Although it is sometimes necessary to use Monte Carlo techniques for generating the statistics, under certain conditions approximate analytic expressions can be found by direct integration.

#### A. First-Order Estimate

Assuming that the random variables  $(\alpha, a, f_1, f_2, \dots, f_n)$  are independent, the variance of the first-order estimate at the  $n^{\text{th}}$  range bin,  $R_n$ , may be

written as,

$$\text{var}(R_n) = E(R_n^2) - E^2(R_n) \quad (19)$$

where

$$E(R_n^2) = c^2 Z_m^{2b} (\sigma_a^2 + a_m^2) E_\alpha \left[ E_{f_n} (f_n^{2b} e^{\alpha p_1 Z_{mn}^\beta f_n^\beta}) \prod_{i=1}^{n-1} E_{f_i} (e^{2\alpha p_1 Z_{mi}^\beta f_i^\beta}) \right] \quad (20)$$

$$E(R_n) = c Z_m^b a_m E_\alpha \left[ E_{f_n} (f_n^b e^{\alpha p_1 Z_{mn}^\beta f_n^\beta / 2}) \prod_{i=1}^{n-1} E_{f_i} (e^{\alpha p_1 Z_{mi}^\beta f_i^\beta}) \right] \quad (21)$$

and

$$E(a) = a_m, \quad \text{var}(a^2) = \sigma_a^2$$

$$p_1 = 0.2 s c^\beta b \ln 10$$

For large  $N$ , the expectation with respect to  $f_i$  ( $i = 1, 2, \dots, n$ ) may be approximated by the first-order term of a saddle-point integration. Because  $\beta$  is usually near 1, an alternative procedure may be used that yields a somewhat more accurate result. To evaluate the expectation with respect to  $f_i$  ( $i = 1, 2, \dots, n-1$ ), the term  $\exp(\Omega f_i^\beta)$  (where  $\Omega = p_1 \alpha Z_{mi}^\beta$  is treated as a constant) is expanded about  $\beta = 1$ ; for the integration over  $f_n$ ,  $\exp(\Omega f_n^\beta)$  is expanded about  $\Omega f_n^\beta = 0$ . For the final integration over  $\alpha$ , a normal distribution is assumed with mean  $\alpha_m$  and variance  $\sigma_\alpha^2$ .

Neglecting terms involving powers of  $\alpha$  greater than 2, the first and second moments of the first-order rain rate may be written

$$E(R_n^2) \approx c^2 Z_m^{2b} (\sigma_a^2 + a_m^2) \sum_{i=0}^2 K_i H_i e^{-(\alpha_m^2 p - q^2/p)} \quad (22)$$



$$E(R_n) \approx c Z_m^b a_m \sum_{i=0}^2 K_i' H_i' e^{-(\alpha_m^2 p - q'^2/p)} \quad (23)$$

where

$$p = 1/2 \sigma_\alpha^2$$

$$q = p_1 \sum_{i=1}^{n-1} Z_{mi}^\beta / (1 + \delta_o) + p \alpha_m$$

$$K_i = \sum_{k=0}^i a_k b_{i-k}$$

$$a_k = \frac{(p_1 Z_{mn}^\beta / (1 + \delta_o))^k \Gamma[(2 - \delta_o) b + N + k\beta]}{(N - 1)! k! N^{(2 - \delta_o) b + k\beta}}$$

$$b_o = 1$$

$$b_1 = (2 - \delta_o) D_1 \sum_{i=1}^{n-1} Z_{mi}^\beta$$

$$b_2 = (2 - \delta_o)^2 (D_1^2 \sum_{i=1}^{n-1} \sum_{j \neq i} Z_{mi}^\beta Z_{mj}^\beta + D_2 \sum_{i=1}^{n-1} Z_{mi}^{2\beta})$$

where

$$D_1 = p_1 (\beta - 1) (Q - \ln N)$$

$$D_2 = p_1^2 (\beta - 1)/N$$

$$Q = 1 + 1/2 + \dots + 1/N = 0.5772157$$



and

$$H_i(q, p) = \begin{cases} 1 & i = 0 \\ q/p & i = 1 \\ (1 + 2q^2/p)/2p & i = 2 \end{cases}$$

$$H'_i = H_i(q', p)$$

For the foregoing unprimed quantities,  $\delta_0 = 0$ ; the primed quantities are given by the same expressions, but with  $\delta_0 = 1$ .

## B. Second-Order Approximation

The rain-rate estimate for the second-order approximation may be written in the form:

$$R_n = ca Z_{mn}^\beta f_n^b \exp \left\{ p_1 \alpha \left[ \sum_{i=1}^n \epsilon_i Z_{mi}^\beta f_i^\beta \exp \left( p_2 \alpha \sum_{j=1}^i \epsilon_j Z_{mj}^\beta f_j^\beta \right) \right] \right\} \quad (24)$$

where,

$$p_2 = 0.2 s \beta c^\beta \ln 10$$

thus,

$$E(R_n^2) = c^2 (\sigma_a^2 + a_m^2) Z_{mn}^{2b} \times \\ \times E_\alpha \left\{ \prod_{i=1}^n E_{fi} (f_n^{2b} \exp [2\alpha p_1 \sum_{i=1}^n \epsilon_i Z_{mi}^\beta f_i^\beta \exp (p_2 \alpha \sum_{j=1}^i \epsilon_j Z_{mj}^\beta f_j^\beta)]) \right\} \quad (25)$$

and

$$E(R_n) = c a_m Z_{mn}^b \times \\ \times E_\alpha \left\{ \prod_{i=1}^n E_{fi} (f_n^b \exp [\alpha p_1 \sum_{i=1}^n \epsilon_i Z_{mi}^\beta f_i^\beta \exp (p_2 \alpha \sum_{j=1}^i \epsilon_j Z_{mj}^\beta f_j^\beta)]) \right\} \quad (26)$$

With respect to the independent random variables  $(f_1, f_2, \dots, f_n)$ , the expectation in equation 25 may be written as

$$\frac{N^{Nn}}{[(N-1)!]^n} \iiint_{x_i = -\infty}^{\infty} \dots \int e^{-N \sum_{i=1}^n (x_i - \ln x_i)} x_n^{2b} \exp \left[ 2\alpha p_1 \sum_{i=1}^n \epsilon_i Z_{mi}^\beta x_i^\beta \exp(\alpha p_2 \sum_{j=1}^i \epsilon_j Z_{mj}^\beta f_j^\beta) \right] \times dx_1 dx_2 \dots dx_n$$

For large  $N$ , the saddle points occur at approximately  $x_i = 1$  ( $i = 1, 2, \dots, n$ ). Evaluation of the integral results in the asymptotic approximation [Gradshteyn and Ryzhik, 1965; Felsen and Marcuvitz, 1973]

$$\prod_{i=1}^n E_{fi}(\cdot) \approx \exp \left[ 2\alpha p_1 \sum_{i=1}^n \epsilon_i Z_{mi}^\beta e^{\alpha p_2 \sum_{j=1}^i \epsilon_j Z_{mj}^\beta} \right] / [1 + 1/12N + 1/288N^2]^n$$

An identical procedure is used for equation 26. Taking the expectation with respect to  $\alpha$ , letting  $u = p(\alpha - \alpha_m)$ , and combining the previous results gives

$$E(R_n^2) \approx c^2 (\sigma_a^2 + a_m^2) Z_{mn}^{2b} \int_{-\infty}^{\infty} e^{-u^2} g(u) du / \sqrt{\pi} (1 + 1/12N + 1/288N^2)^n \quad (27)$$

$$E(R_n) \approx c a_m Z_{mn}^b \int_{-\infty}^{\infty} e^{-u^2} g'(u) du / \sqrt{\pi} (1 + 1/12N + 1/288N^2)^n \quad (28)$$

where

$$g(u) = \exp \left[ 2p_1 (u/\sqrt{p} + \alpha_m) \sum_{i=1}^n \epsilon_i Z_{mi}^\beta e^{p_2 (u/\sqrt{p} + \alpha_m) \sum_{j=1}^i \epsilon_j Z_{mj}^\beta} \right]$$

$$g'(u) = \sqrt{g(u)}$$

The remaining integrals were computed by means of a Gaussian-Hermite quadrature.

### C. Hitschfeld-Bordan Estimate

A problem arises in calculating the statistics of the Hitschfeld-Bordan estimate because the rain-rate estimate becomes imaginary when

$$K\beta c^\beta \sum_{i=1}^n \epsilon_i f_i^\beta Z_{m,i}^\beta > 1$$

To circumvent the problem, a Monte Carlo technique is used. As before, the estimate is characterized by its mean and variance. In addition, however, it is necessary to define a failure rate as the percentage of times the estimated rain rate is imaginary.

## IV. RESULTS AND DISCUSSION

The results of the three rain-rate estimates for a radar operating at 8.75 GHz are shown in the Figures. The  $Z_t$ -R,  $k$ - $Z_t$  relations were derived from the tabulated values of Medhurst [1965] and Stephens [1961] and are approximated by  $Z_t = 307.1 R^{1.54}$ ,  $k = 5.5 \times 10^{-5} Z_t^{0.84}$ . For reasons of economy, both the mean and variance are plotted on the same graphs as functions of range for values of N of 100 and 1000. The "X" and "O" labeled points represent, respectively, the variance and mean of the rain-rate estimates (normalized to the true value) as computed by the Monte Carlo method. Where present, the solid and dashed curves represent the variance and mean as calculated from the approximate analytic expressions found in Section II. Unless otherwise noted,

an increase in  $N$  results in a decrease in variance and an increase in mean.

The change in the latter quantity, however, is not significant.

Comparison of analytic and Monte Carlo evaluations of the statistics are shown in Figures 1 and 4, and 2 and 5 for the first- and second-order estimates, respectively. In addition to the qualitative insight that may be gained, the analytic results eliminate the spurious fluctuations that arise from the Monte Carlo generation at small values of variance. It can be seen that the approximation for the second-order estimate for  $N = 100$  is somewhat crude, implying that higher order terms are needed in the multiple-integral saddle-point technique.

The remaining graphs presented here were found by computing the sample mean and variance of 1000 simulations per range bin. To achieve adequate plotting resolution, the upper limit for the normalized variance was set at 2, which resulted in zero failure rates for the Hitschfeld-Bordan estimates that were plotted.

The most prominent features of the results follow:

- In the absence of radar calibration errors and for low rain rates and small  $\sigma_a^2$ ,  $\sigma_\alpha^2$ , the Hitschfeld-Bordan estimate is relatively unbiased—a consequence of the fact that the true reflectivity factor has been exactly expressed as a function of the measured reflectivity factor,  $Z_m$ .
- Even in the absence of calibration errors, the first- and second-order estimates are biased by an amount proportional to the total attenuation.



The use of higher order estimates tends to decrease the offset error (Figures 1, 2, 4, and 5).

- In general, the error variances are largest and the sensitivity to calibration errors are most pronounced when a Hitschfeld-Bordan estimate is used (Figures 3 and 6). Because of the appearance of the  $ca$  and  $c^\beta \alpha$  factors in all the estimates, the effects of calibration errors are dependent on the statistics of  $a$  and  $\alpha$ . Note that the variances of these factors are  $c^2 \sigma_a^2$ ,  $c^{2\beta} \sigma_\alpha^2$ . If  $c < 1$  (i.e., the radar calibration constant is larger than the true value), a smaller error variance is expected than if the calibration constant were underpredicted ( $c > 1$ ). This type of behavior is indicated in Figures 7 through 12.
- The rain-rate estimate for nonattenuating radar at the  $n^{\text{th}}$  range bin can be written as

$$R_n = ca Z_{\text{tn}}^b f_n$$

The two characteristics that distinguish  $R_n$  from the attenuated estimates are:

- a. The calibration error enters into the estimate only through the multiplicative factor,  $c$ .
- b. The estimate is independent of the preceding range bins.

A comparison between attenuating and nonattenuating radars can be obtained by noting that the mean and the variance of the latter are independent of range

(assuming large signal-to-noise ratios) and nearly equal to the values of the attenuated radar in the first range bin. The vertical deflection from this constant mean and variance indicates the additional errors that arise in using an attenuated frequency in conjunction with a particular reconstruction technique.

## V. CONCLUSIONS

Three attenuation correction procedures have been considered that have led to the corresponding rain-rate estimates. For all estimates, an attempt has been made to include the effects of fluctuations in the averaged radar return power and the variability in the  $Z$ - $R$ ,  $k$ - $Z$  relations. For several combinations of meteorological conditions and radar parameters, the mean and variance were computed and plotted as a function of range.

The results indicate that, in addition to the Hitschfeld-Bordan procedure, alternative correction schemes exist that lead to rain-rate estimates with smaller error variances and a decreased sensitivity to positive offsets in the radar calibration constant. On the other hand, corrections such as the first-order algorithm may lead to estimates that significantly underpredict the true value of rain rate.

Although it is impossible to choose the "best" correction technique without specifying the radar parameters and performance criteria, a few general characteristics of the various order estimates can be seen.

For a well-calibrated radar using relatively accurate  $k$ - $Z$ ,  $Z$ - $R$  relations, higher order correction techniques may be employed. When the possibility of



significant errors exist, the lower order estimates must be used to avoid large error variances and large positive biases of the estimate.

It is reasonable to assume that, given a radar design and performance criteria, there will be some  $k^{\text{th}}$  order estimate (with  $k$  finite) that optimizes the radar performance. For the criteria given, this estimate will represent the best compromise between the extreme cases of no attenuation correction and the attempt at exact correction.

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The author thanks Mohsen Khalil and Jerome Eckermann whose contributions are evident throughout this paper.

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## APPENDIX A

The  $n^{\text{th}}$  order approximation to  $Z_t$  is given by

$${}^n Z(r) = Z_m(r) \exp \left( K \int_0^r {}^{n-1} Z^\beta(r) dr \right) \quad (\text{A1})$$

subject to the conditions that

$${}^0 Z(r) = Z_m(r)$$

$${}^n Z(0) = Z_m(0) = Z_t(0)$$

Taking the logarithm of both sides of equation A1, differentiating with respect to range, and letting  $u_k = {}^k Z^{-\beta}$  yields the difference differential equation,

$$du_n/dr = -\beta u_n (f(r) + K/u_{n-1}) \quad (\text{A2})$$

where

$$f(r) = d(\ln Z_m)/dr = (1/Z_m) dZ_m/dr$$

From the definitions of  ${}^n Z(r)$  and  $Z_t(r)$  and by induction, the following two conditions can be shown to hold for all values of  $r$ :

$${}^n Z(r) \geq {}^{n-1} Z(r)$$

$$Z_t(r) \geq {}^n Z(r)$$

To show that  $\lim_{n \rightarrow \infty} {}^n Z(r) = Z_t(r)$ , consider the series and an upper bound,

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N ({}^n Z(r) - {}^{n-1} Z(r)) = \lim_{N \rightarrow \infty} ({}^N Z(r) - Z_m(r)) \leq (Z_t(r) - Z_m(r))$$

From the solution of the differential equation (equation 10), it can be seen that  $Z_t(r)$  is bounded if  $K\beta \int_0^r Z_m^\beta(r) dr < 1$ . Therefore, under this condition, the series converges, implying that

$$\lim_{n \rightarrow \infty} ({}^n Z - {}^{n-1} Z) = 0$$

or, equivalently,

$$\lim_{n \rightarrow \infty} (u_n / u_{n-1}) = \lim_{n \rightarrow \infty} ({}^{n-1} Z / {}^n Z)^\beta = 1$$

In the limit of large  $n$ , the difference differential equation becomes

$$du_n/dr = -\beta f(r) u_n - \beta K \quad (A3)$$

subject to the condition,  ${}^n Z(0) = Z_m(0)$ . But this differential equation and initial condition are identical to those satisfied by  $Z_t$  (equation 9), so that

$$\lim_{n \rightarrow \infty} {}^n Z(r) = Z_t(r)$$



## APPENDIX B

The generalization of the estimates to the case of finite signal-to-noise ratios is straightforward. The power available at the receiver is now the sum of  $N$  independent samples of return power, plus receiver noise. Assuming that the noise power,  $p_N$ , is statistically independent of the backscattered power,  $p(r)$ , but follows the same distribution, the power from the receiver can be written as [Stogryn, 1975]

$$\hat{P} = (P(r) + P_N) f \quad (B1)$$

As before, defining the measured reflectivity factor by

$$Z_m = P(r)r^2/C$$

where

$$C = C_{\text{true}} (1 + E)$$

then,

$$\hat{Z}_m = (P(r) + P_N) fr^2/C_{\text{true}} (1 + E)$$

or at the  $i^{\text{th}}$  range bin,

$$\hat{Z}_{mi} = Z_{mi} f_i (1 + P_N/P_i(r))/(1 + E) \quad (B2)$$

Comparison of equation B2 with equation 15 of the main report shows that the effect of finite signal to noise is to introduce an additional term  $(1 + p_n/p(r))$  into the definition of the measured reflectivity factor. Proceeding as before,



$Z_t$  is expressed as a function of  $Z_m$  by various order approximations that, in turn, lead to three estimates of rain rate analogous to those of equations 16 through 18.

For example, the first-order rain rate becomes

$$^1R_n = ca Z_{mn}^b f_n^b (1 + P_N/P_n(r))^b \exp\left(K\beta c^\beta \sum_{i=1}^n \epsilon_i Z_{mi}^{\beta} f_i^{\beta} (1 + P_N/P_i(r))^{\beta}\right) \quad (B3)$$

## REFERENCE

Stogryn, A. (1975), Error Analysis of the Goldhirsch-Katz Method of Rainfall Rate Determination by the Use of a Two Frequency Radar, Report 1833 TR-4, Aerojet ElectroSystems Co., Azusa, California.

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Attenuation Correction ( $R = 25$  mm/hr,  $\sigma_a = a/20$ ,  $\sigma_\alpha = \alpha/10$ ,  $c = 2/3$ ).

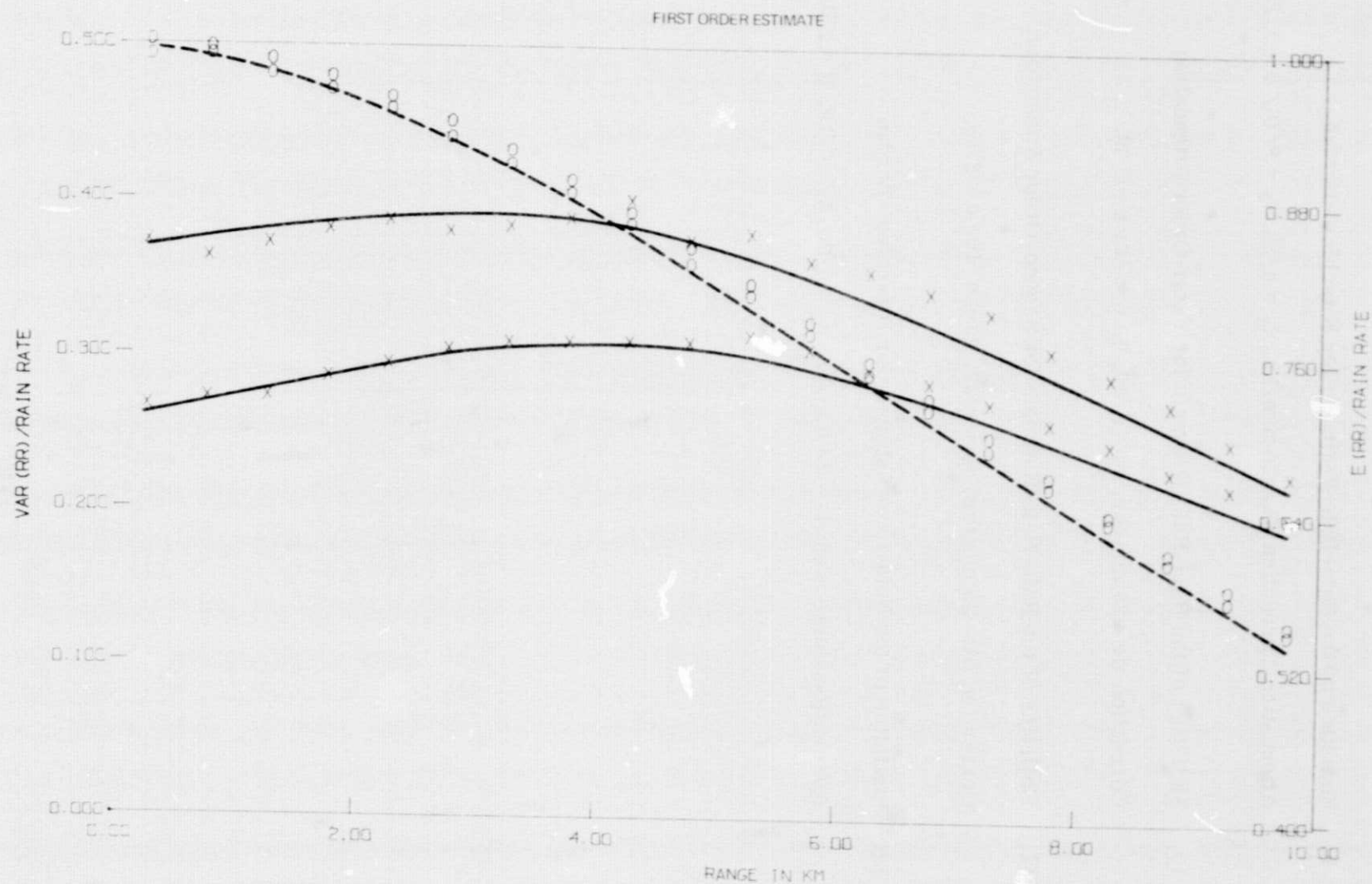


Figure 1. Statistics of the Rain-Rate Estimate for First-Order Attenuation Correction ( $R = 25$  mm/hr,  $\sigma_a = a/10$ ,  $\sigma_\alpha = a/5$ ,  $c \approx 1$ ).

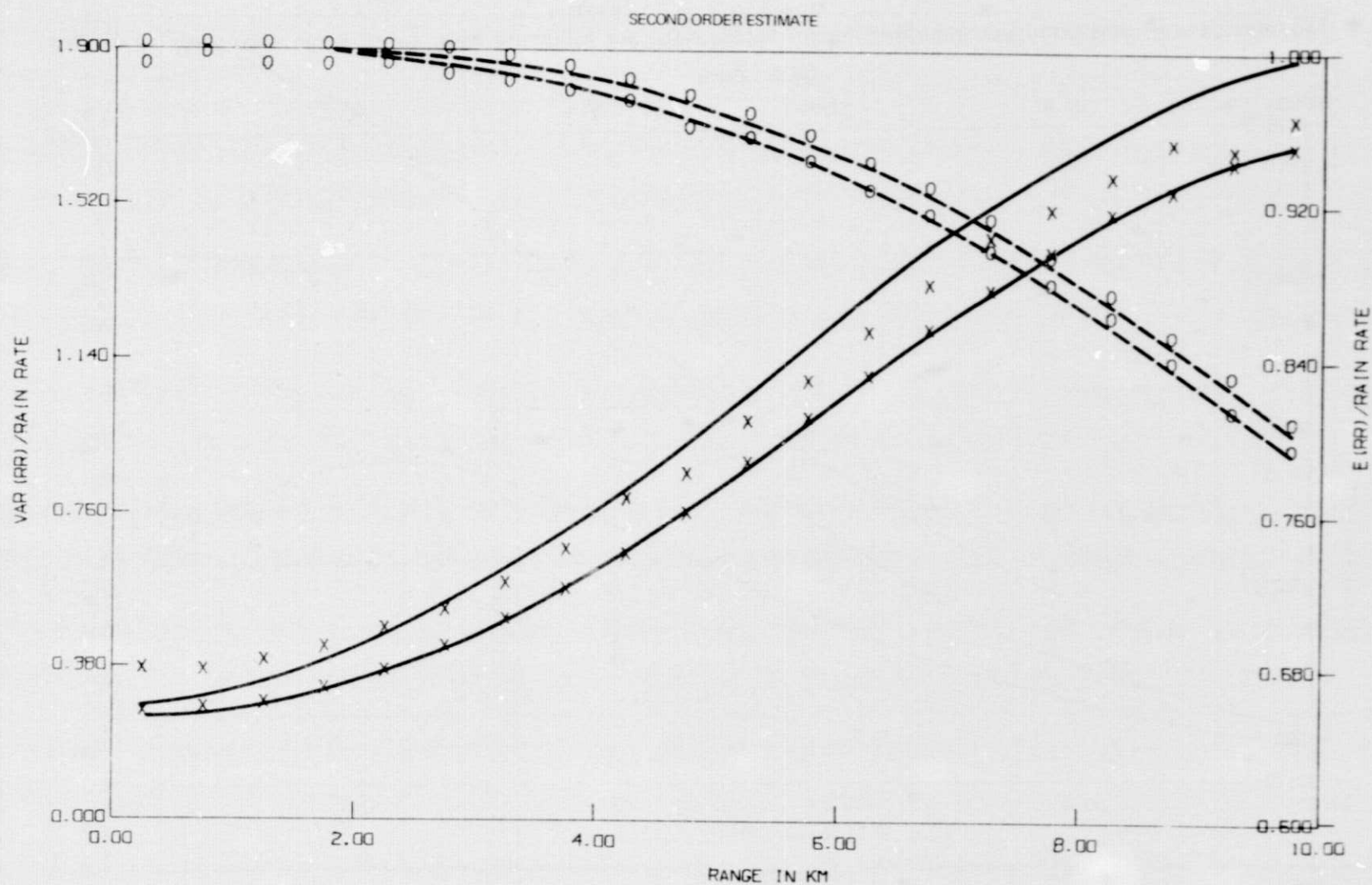


Figure 2. Statistics of the Rain-Rate Estimate for Second-Order Attenuation Correction ( $R = 25$  mm/hr,  $\sigma_a = a/10$ ,  $\sigma_z = a/5$ ,  $c = 1$ ).



HITSCHFELD - BORDAN ESTIMATE

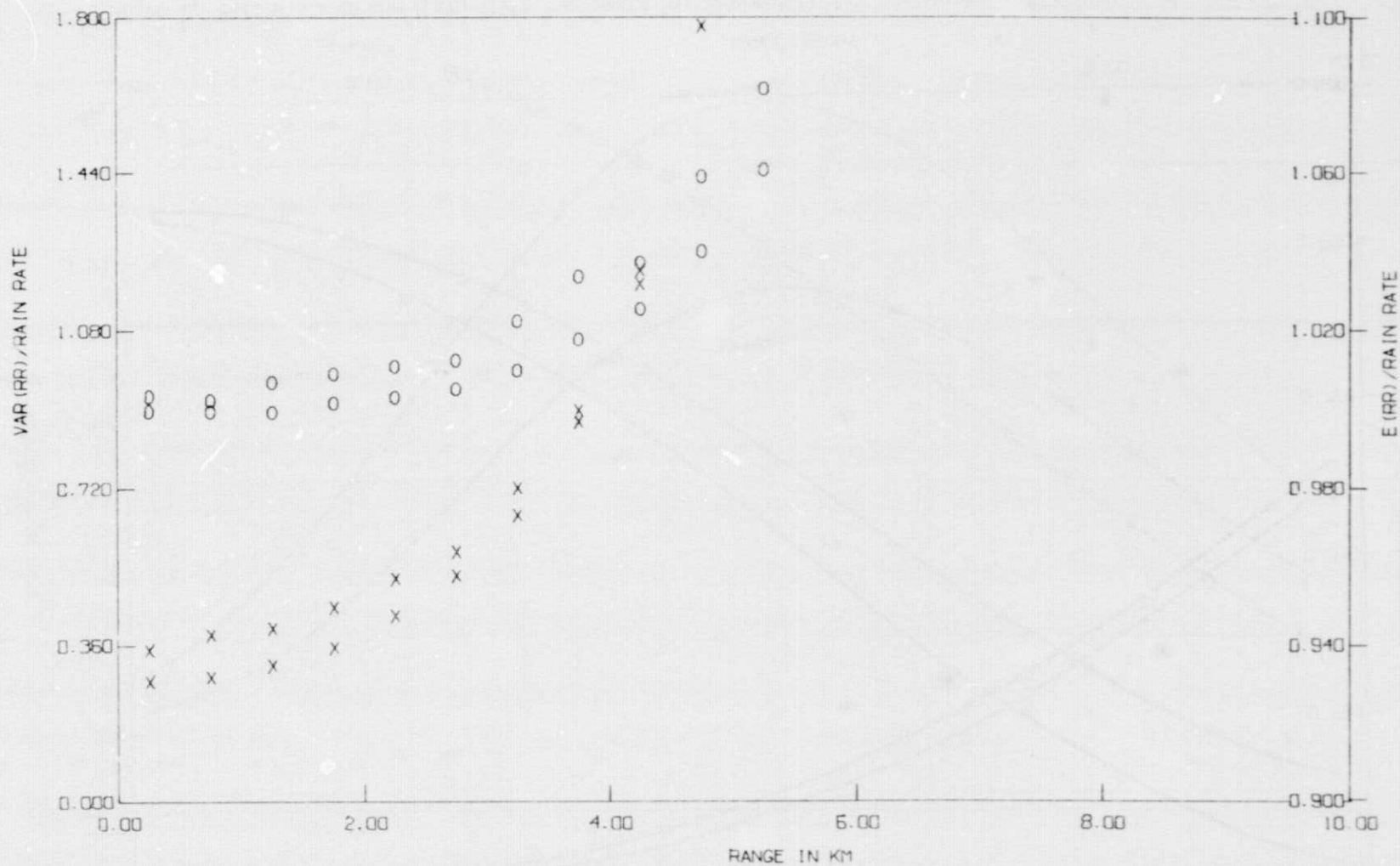


Figure 3. Statistics of the Rain-Rate Estimate for Hitschfeld-Bordan Attenuation Correction ( $R = 25$  mm/hr,  $\sigma_a = a/10$ ,  $\sigma_a = a/5$ ,  $c = 1$ ).

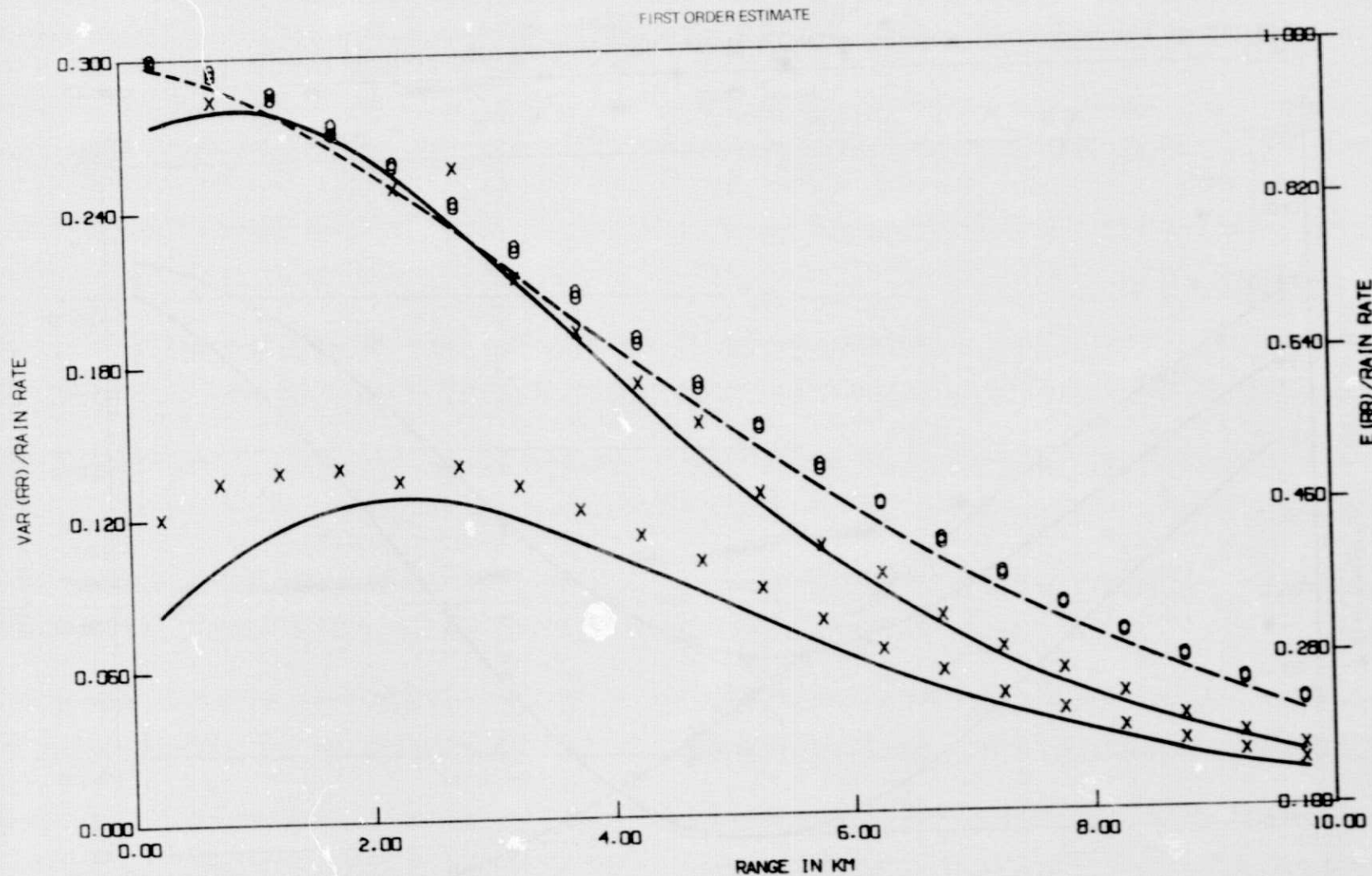


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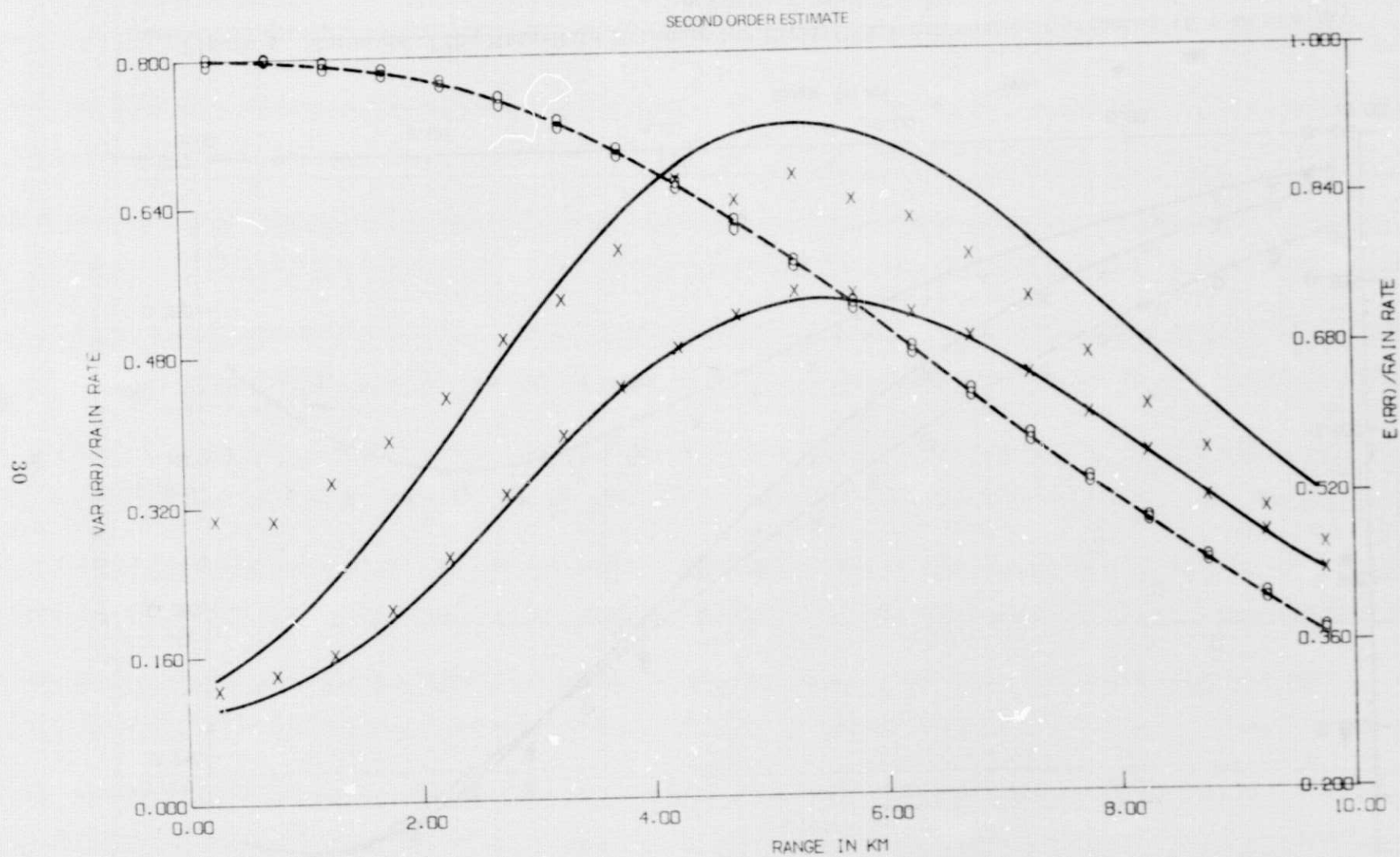


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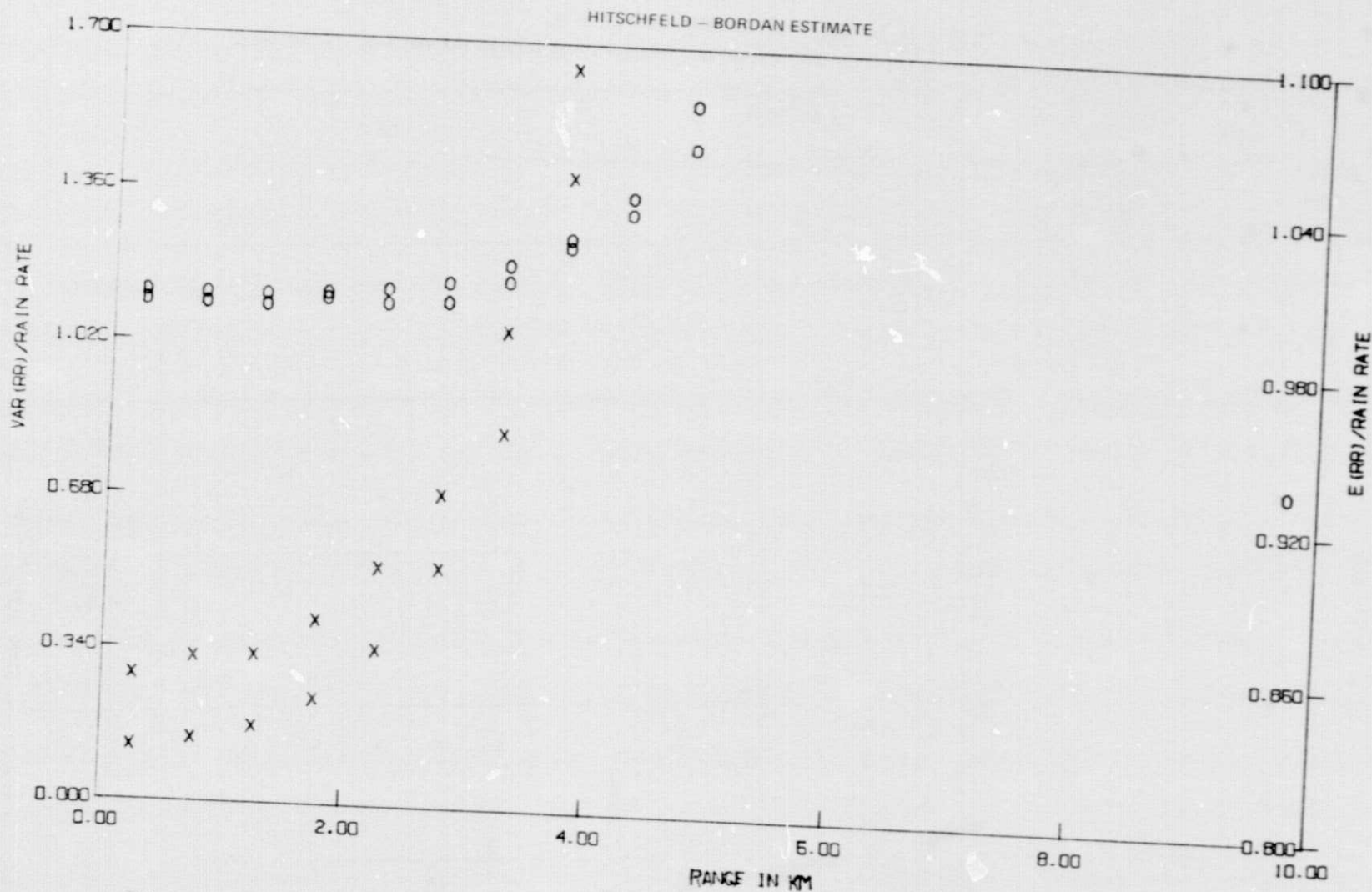


Figure 6. Statistics of the Rain-Rate Estimate for Hitschfeld-Bordan Attenuation Correction  
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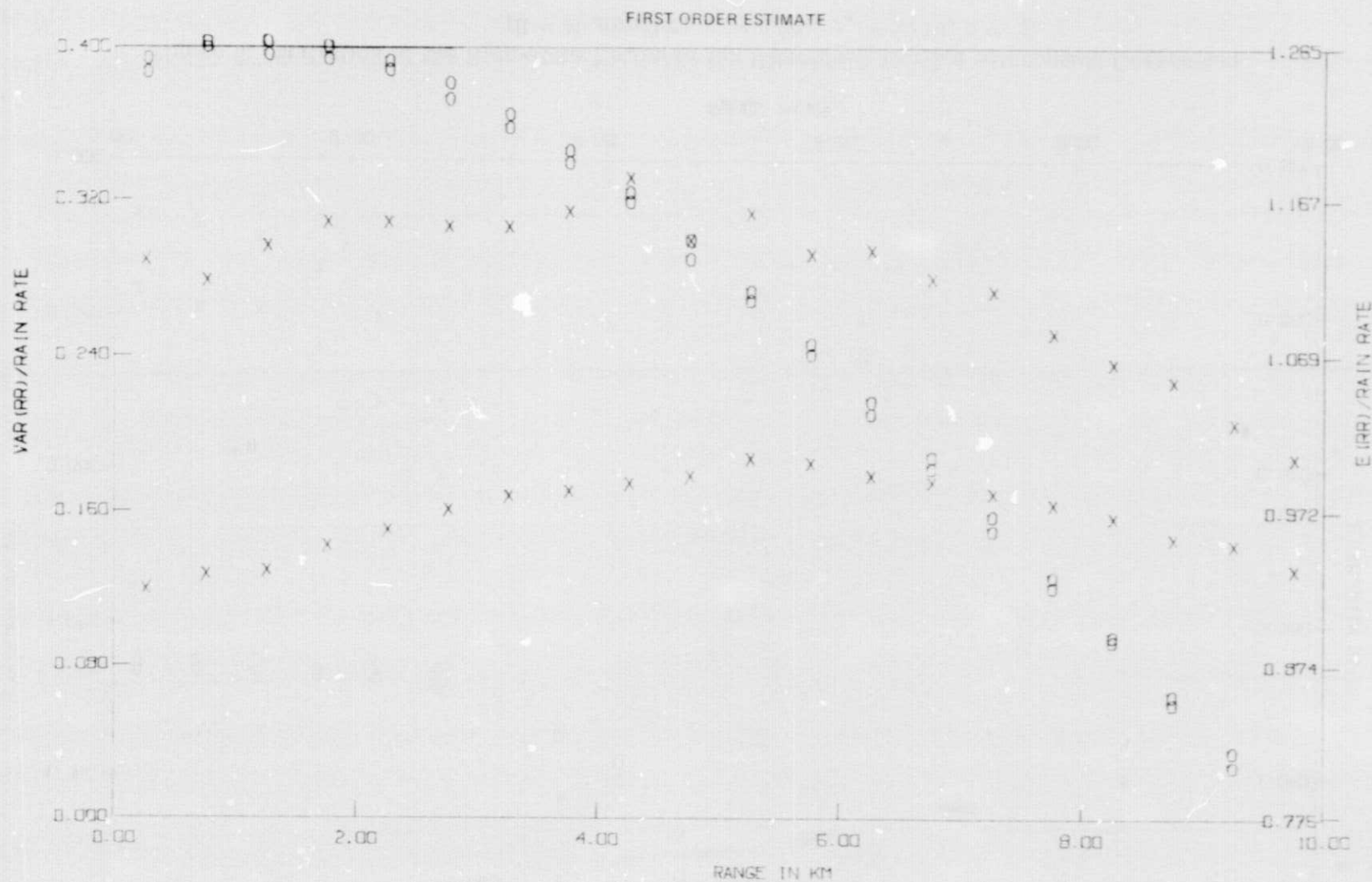


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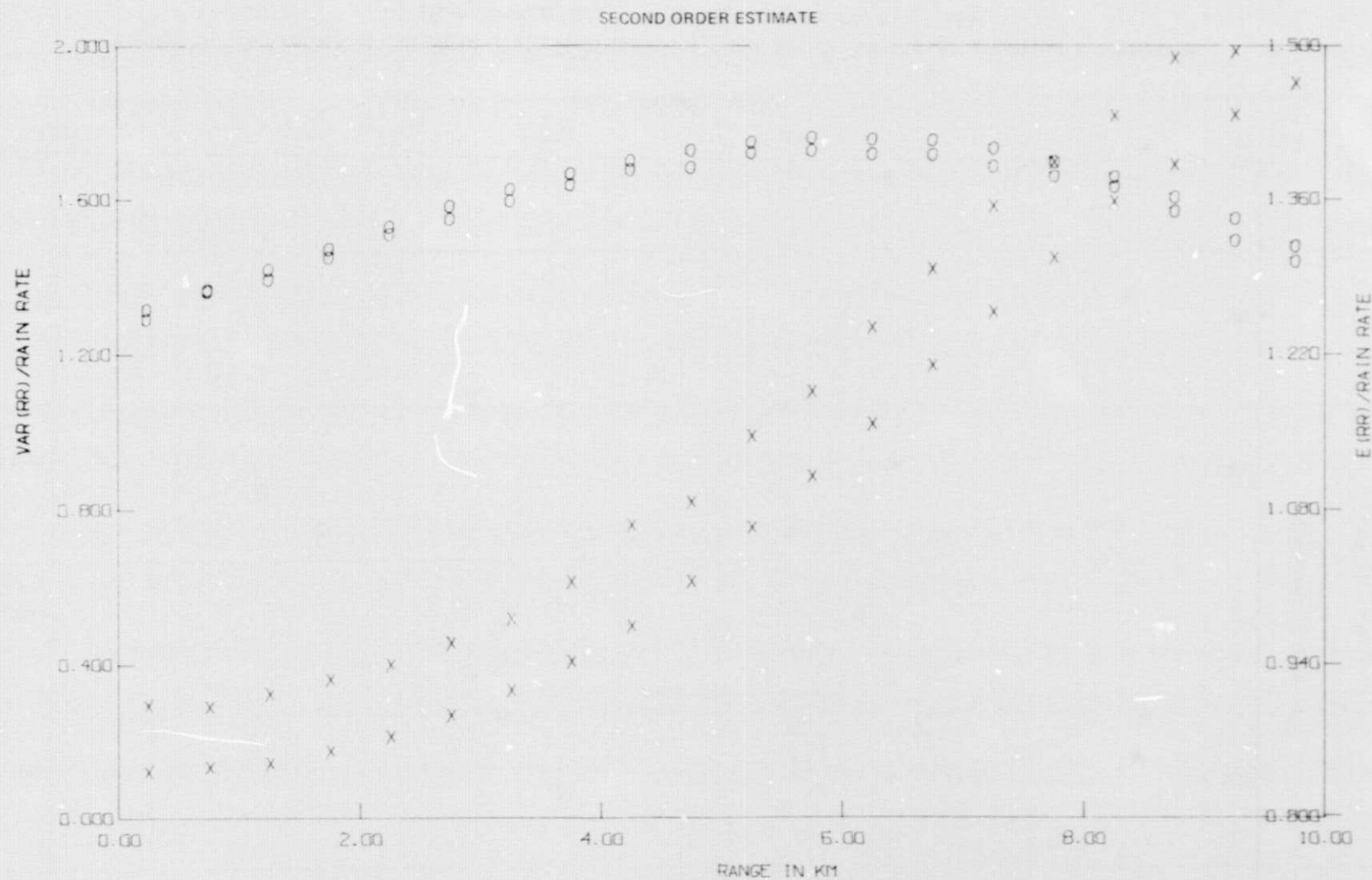


Figure 8. Statistics of the Rain-Rate Estimate for Second-Order Attenuation Correction  
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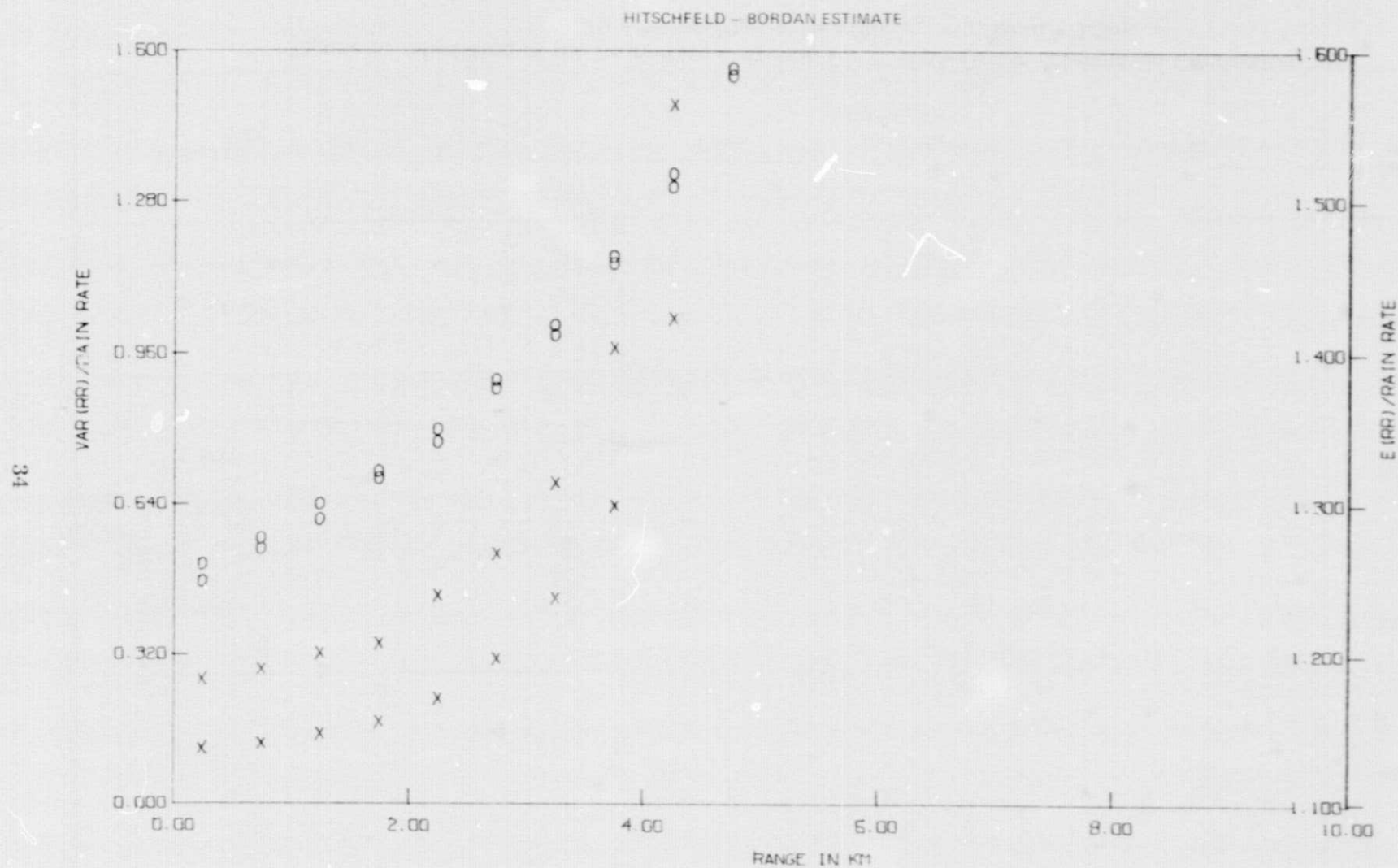


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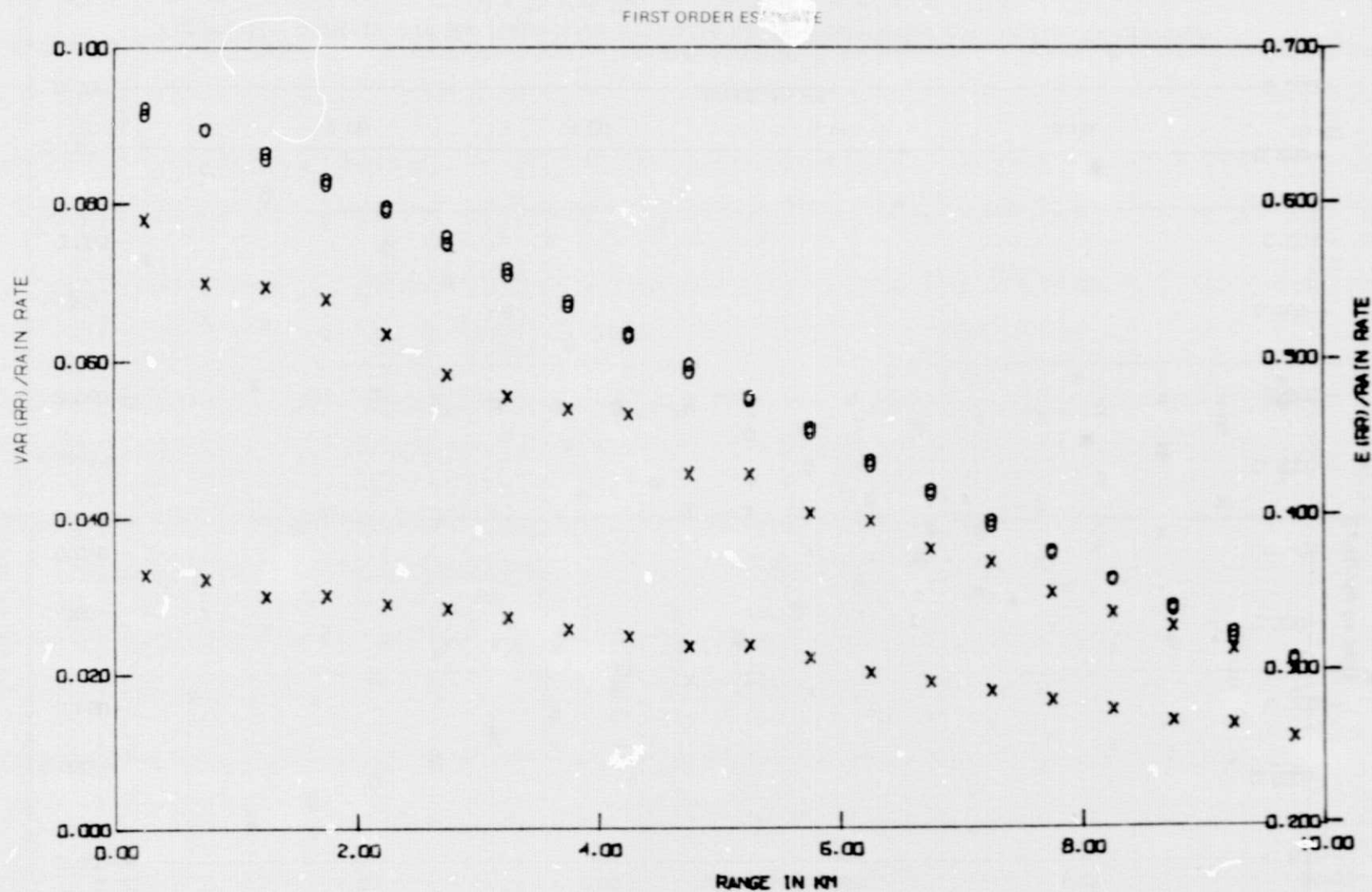


Figure 10. Statistics of the Rain-Rate Estimate for First-Order Attenuation Correction  
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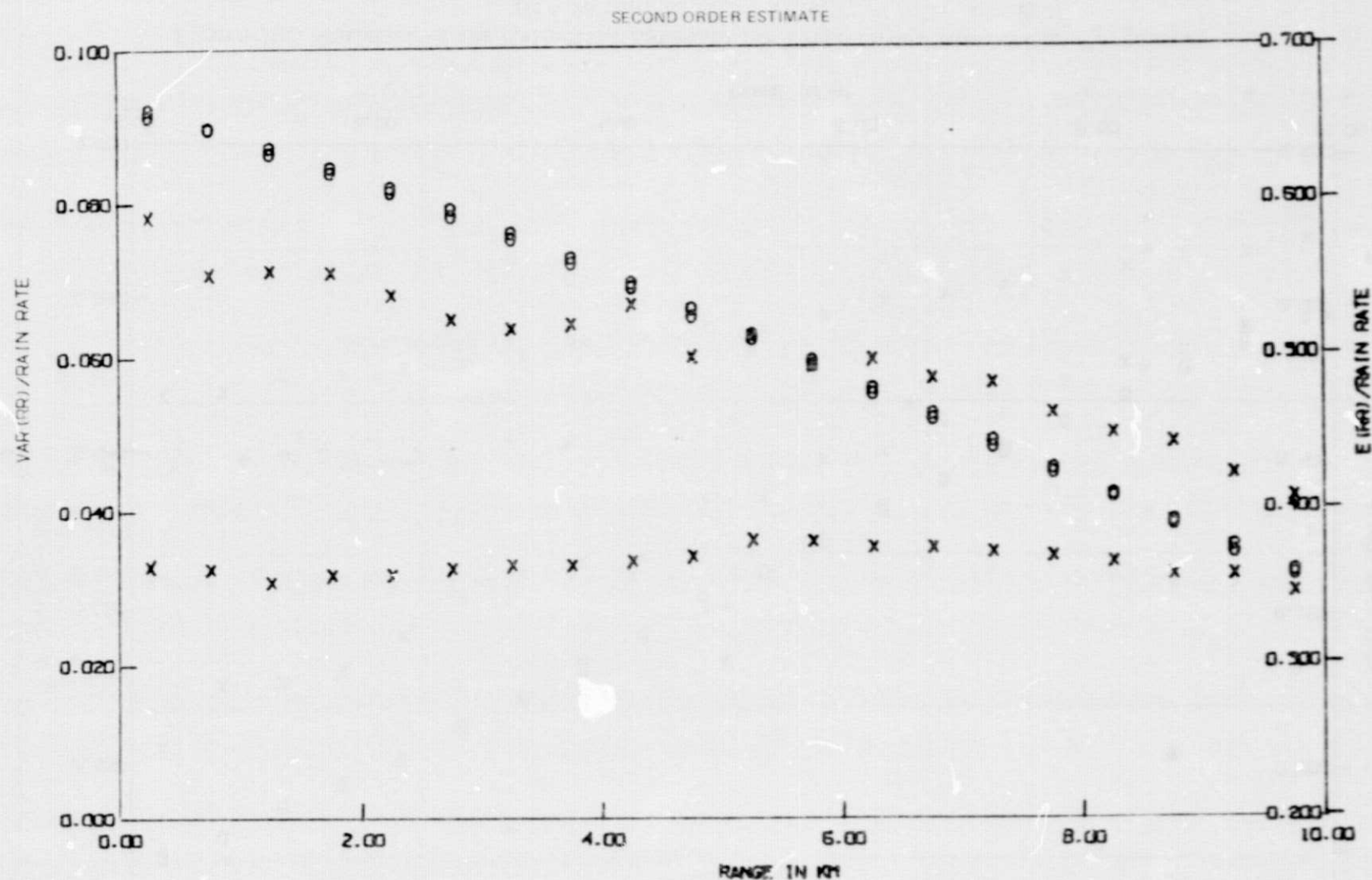


Figure 11. Statistics of the Rain-Rate Estimate for Second-Order Attenuation Correction  
 $(R = 25 \text{ mm/hr}, \sigma_a = a/20, \sigma_c = c/10, c = 2/3)$ .

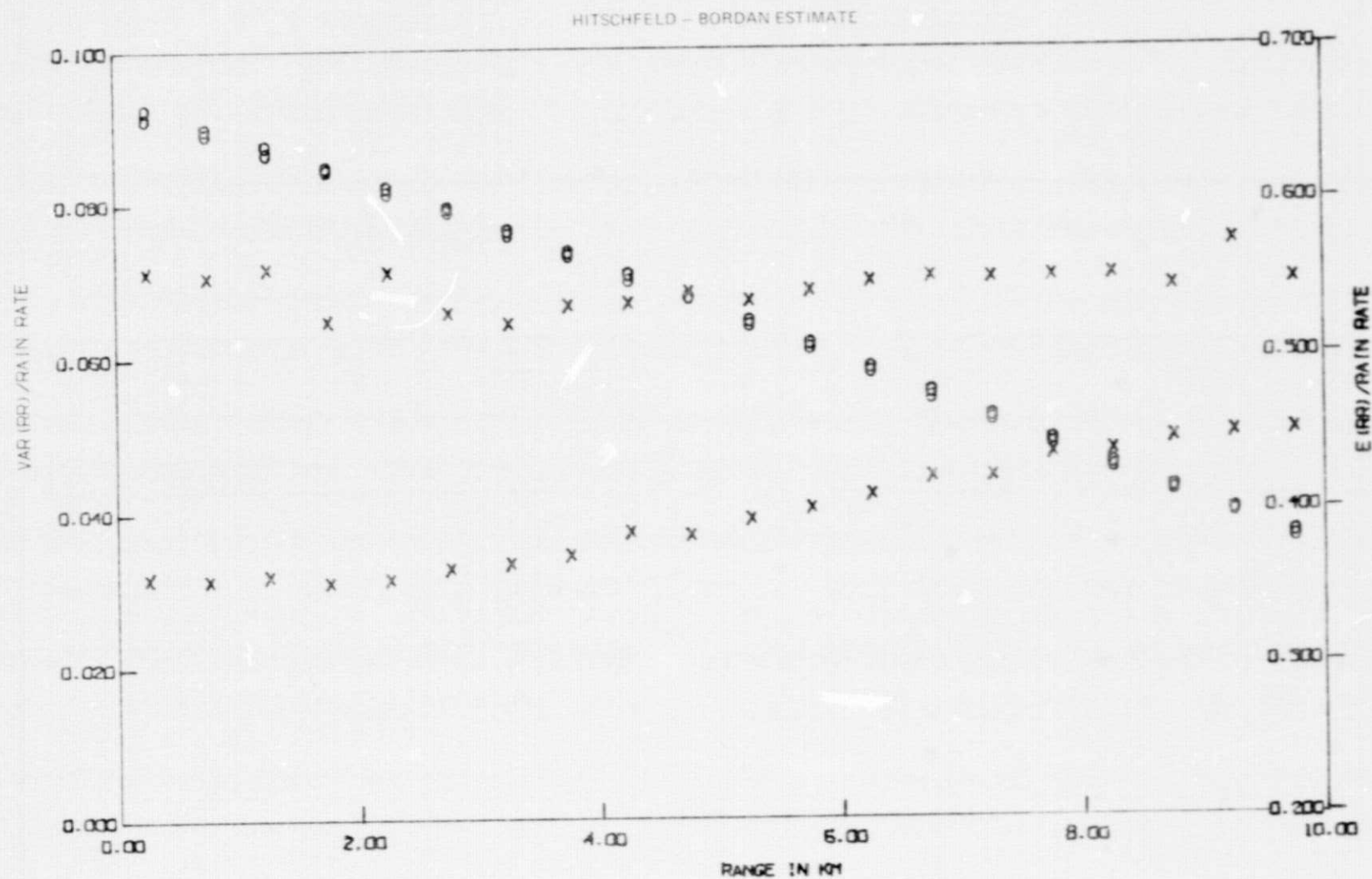


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